The math behind Spot It!

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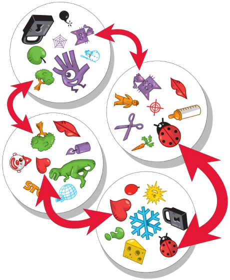


Figure 1: Four cards in a Spot It! deck

The setup of Spot It!

A Spot It! deck has

- **55 cards**,
- 8 symbols on each card,
- 1 shared symbol between any 2 cards.

Each player starts off with one card in their hand, and the remaining cards are stacked in the center. Each turn consists of the players attempting to spot a match between an element in their card and the top card in the stack. If they find a match first, they take the card in the middle as their new card revealing a new card in the middle stack. The goal of the game is to have collected the most cards by the end. This game is secretly quite similar to the other pattern-finding game Set.

Question

A Spot It! deck "should" have **57 cards**. Why?

Question

What are the two missing cards?

The projective plane

Definition

The projective plane over a field F is a 3-dimensional space consisting of lines $Ax + By + Cz = 0$, where A, B, C belong to F. In projective space, scalar multiples are equivalent, meaning that $[A : B : C]$ is equivalent to $[kA : kB : kC]$, as $Ax + By + Cz = 0$ is the same line as $kAx + kBy + kCz = 0$.

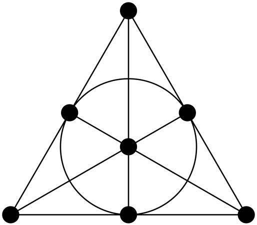


Figure 2: The Fano plane is the projective plane over $\mathbb{F}_2$ (residues mod 2 that you can add and multiply). There are 7 points and 7 lines. Each pair of lines shares one point in common.

The Fano plane and Spot It!

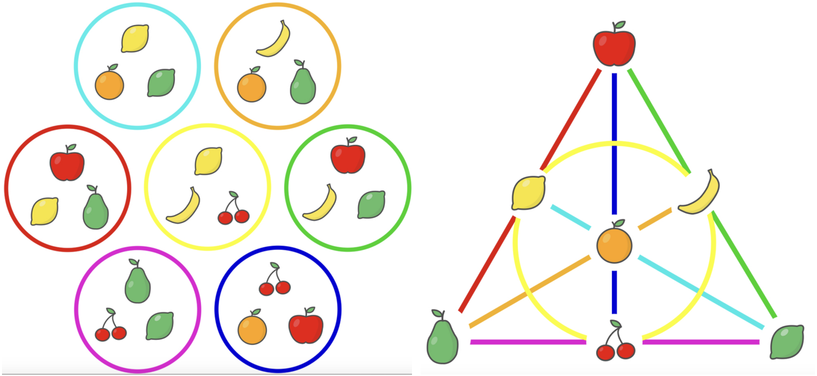


Figure 3: A small Spot It! deck and its corresponding Fano plane. As we can see, each of the 7 cards corresponds to one of the 7 lines in the Fano plane. Just like any line has 3 points on it, any card has 3 symbols on it. Also, just like any two distinct lines in the Fano plane share exactly 1 point in common, each pair of distinct cards has exactly 1 symbol in common.

Formal definition of a field

Definition

A field is a set F together with two binary operations, addition and multiplication, that satisfy the following axioms.

- Associativity of addition and multiplication: $a + (b + c) = a + (b + c)$, $a(bc) = (ab)c$
- Commutativity of addition and multiplication: $a + b = b + a$, $ab = ba$
- Existence of additive identity 0 and multiplicative identity 1: $a + 0 = a$, $a \cdot 1 = a$
- Existence of additive and multiplicative inverses: $a + (-a) = 0$, $aa^{-1} = 1$
- Distributivity of multiplication over addition: $a(b + c) = ab + ac$.

The residues mod p under addition and multiplication mod p form a field. This field is denoted $\mathbb{F}_p$.

Combinatorics of projective planes

Counting the number of points in a projective plane, recall that points in a projective plane are given by triples $[a : b : c]$.

There would be $n^3 - 1$ triples of points in $\mathbb{F}_n$ where not all coordinates are zero, but this value overcounts by a factor of $n - 1$ because each of the $n - 1$ nonzero multiples $[a : b : c]$, $[2a : 2b : 2c]$, etc., all represent the same point in projective space.

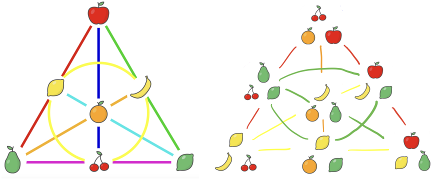
Counting the lines, we note that there are $\binom{n^2+n+1}{2}$ ways to choose 2 points, which determine a line. However, this value overcounts because a line passes through $n + 1$ points, and any choice of 2 of these $n + 1$ points gives the same line. So we must divide by a factor of $\binom{n+1}{2}$, in the end, getting that

$$\frac{\binom{n^2+n+1}{2}}{\binom{n+1}{2}} = n^2 + n + 1.$$

In our Spot It! deck, we have $n = 7$. Since each card stands for a line in projective plane, there should be $7^2 + 7 + 1 = 57$ cards in a Spot It! deck.

The missing Spot It! cards, part 1

In projective space, there is a duality between points in lines, implying that the number of points and lines in a finite projective plane are equal to each other. It also explains why we can represent the small Spot It! deck using the Fano plane in two different ways.



In the small deck, each symbol appears on exactly 3 different cards. However, in the real Spot It! deck, each symbol appears on exactly 8 different cards. But there are missing cards, so some symbols appear less than 8 times. By systematically counting these symbols, we can figure out which symbols belong on the missing Spot It! cards.

The missing Spot It! cards, part 2

Figure 4: the Spot It! cards are arranged like points in the projective plane over $\mathbb{F}_7$. Each line of cards shares exactly one symbol. For example, all the cards in the rightmost column have a bomb.

The missing Spot It! cards, part 3

Here are the missing cards.

Acknowledgements

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